Problem 2.12

Problem 2.7 is about a class of one-dimensional problems that can always be reduced to doing an integral. Here is another. Show that if the net force on a one-dimensional particle depends only on position, $F = F(x)$, then Newton's second law can be solved to find v as a function of x given by

$$v^2 = v_o^2 + \frac{2}{m}\int_{x_o}^{x} F(x')\,dx'. \tag{2.85}$$

[*Hint:* Use the chain rule to prove the following handy relation, which we could call the "$v\,dv/dx$ rule": If you regard v as a function of x, then

$$\dot{v} = v\frac{dv}{dx} = \frac{1}{2}\frac{dv^2}{dx}. \tag{2.86}$$

Use this to rewrite Newton's second law in the separated form $m\,d(v^2) = 2F(x)\,dx$ and then integrate from x_o to x.] Comment on your result for the case that $F(x)$ is actually a constant. (You may recognise your solution as a statement about kinetic energy and work, both of which we shall be discussing in Chapter 4.)

Solution

Suppose there's a particle moving in the x-direction. By Newton's second law,

$$F = ma.$$

If the force depends on position, then

$$\begin{aligned} F(x) &= ma \\ &= m\frac{dv}{dt} \\ &= m\frac{dv}{dx}\frac{dx}{dt} \\ &= m\frac{dv}{dx}v \\ &= \frac{m}{2}\left(2v\frac{dv}{dx}\right) \\ &= \frac{m}{2}\frac{d}{dx}(v^2). \end{aligned}$$

Divide both sides by $m/2$.

$$\frac{2}{m}F(x) = \frac{d}{dx}(v^2)$$

Integrate both sides from x_o to x. Since the integration is definite, no constant of integration is needed.

$$\int_{x_\text{o}}^{x} \frac{2}{m} F(x')\,dx' = \int_{x_\text{o}}^{x} \frac{d}{dx'}(v^2)\,dx'$$

$$\begin{aligned} \frac{2}{m}\int_{x_\text{o}}^{x} F(x')\,dx' &= v^2\Big|_{x_\text{o}}^{x} \\ &= [v(x)]^2 - [v(x_\text{o})]^2 \\ &= v^2 - v_\text{o}^2 \end{aligned}$$

Therefore,

$$v^2 = v_\text{o}^2 + \frac{2}{m}\int_{x_\text{o}}^{x} F(x')\,dx'.$$

This can also be written as

$$\int_{x_\text{o}}^{x} F(x')\,dx' = \frac{1}{2}mv^2 - \frac{1}{2}mv_\text{o}^2$$

$$W_\text{total} = \Delta\text{KE},$$

which is the work-energy theorem. If $F(x)$ is a constant, then this formula simplifies to a known kinematic formula.

$$\begin{aligned} v^2 &= v_\text{o}^2 + \frac{2}{m}\int_{x_\text{o}}^{x} F_\text{o}\,dx' \\ &= v_\text{o}^2 + \frac{2}{m}F_\text{o}(x - x_\text{o}) \\ &= v_\text{o}^2 + 2a(x - x_\text{o}). \end{aligned}$$